ASYMPTOTIC BEHAVIOR OF THE CONVOLUTION OF A PAIR OF MEASURES(1)

BY
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1. **Introduction.** Let the time parameter set T be given either as the discrete set $T=\{0, 1, 2, \ldots\}$ or as the continuous set $T=[0, \infty)$. Consider random counting measures N_t, $t \in T$, defined on the Borel sets of d-dimensional Euclidean space R^d. Here $N_t(A)$ denotes the "number of particles" in A at time t. Set $N=N_0$, let ν_t, $t \in T$, be the measures defined by $\nu_t(A)=EN_t(A)$, and set $\nu=\nu_0$. The temporal structure of N_t is determined by letting the particles in R^d at time zero be translated independently according to stochastic processes isomorphic to a fixed stochastic process Y_t, $t \in T$. Let μ_t denote the distribution of Y_t. Then

$$\nu_t(A) = EN_t(A) = (\nu * \mu_t)(A)$$

and

$$E[N_t(A) \mid N] = (N * \mu_t)(A).$$

Extending and correcting results of Dobrushin [1], the author in [2] studied the asymptotic behavior as $t \to \infty$ of $(\nu * \mu_t)(A)$ and $(N * \mu_t)(A)$.

Let dt, $t \in T$, refer to counting measure and Lebesgue measure in the discrete and continuous cases respectively. Let the random measure N^* be defined by

$$N^* = \int_0^\infty N_t\, dt.$$

Then $N^*(A)$ denotes the total occupation time of A by all the particles. Let μ denote the measure defined by

$$\mu = \int_0^\infty \mu_t\, dt.$$

Then EN^*, the expected total occupation time measure, is given by $EN^*=\nu * \mu$, and $E[N^* \mid N]$ is given by $E[N^* \mid N]=N * \mu$.

In this paper we will present a general study of the asymptotic behavior of $(\nu * \mu)(x+A)$ and $(N * \mu)(x+A)$ as $x \to \infty$ appropriately. The intended application is to the case in which $\mu_{s+t}=\mu_s * \mu_t$, i.e. when Y_t is a random walk or a process with independent increments. Then μ is the multidimensional renewal measure. This application will be presented in [3].

In §2 we give precise definitions and state our results. The proofs are given in §§3 and 4. Many of the arguments used here are refinements of those used in [2]. The

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framework discussed above was intended mainly for motivation, that of the rest of the paper being somewhat more general.

2. **Definitions and statements of results.** Let X denote a d-dimensional closed subgroup of R^d. With no loss of generality, we can assume that X is of the form

$$X = \{x = (x^1, \dots, x^d) \mid x^k \text{ are integers for } d_1 < k \leqq d\}.$$

Set $Z^d = \{x \mid x^k \text{ are integers for } 1 \leqq k \leqq d\}$. If $d_1 = d$, then $X = R^d$ and if $d_1 = 0$, then $X = Z^d$.

Set $\Delta = \{x \in X \mid 0 \leqq x^k < 1 \text{ for } 1 \leqq k \leqq d\}$. Set $U = \{x \in Z^d \mid x^k = 1$ for some k and $x^j = 0$ for $j \neq k\}$. Then U consists of d "unit vectors." For $a > 0$ and $x \in X$, set

$$a \odot x = (ax^1, \dots, ax^{d_1}, x^{d_1+1}, \dots, x^d).$$

Let $| \quad |$ denote Haar measure on X defined as the product of Lebesgue measure on the first d_1 coordinates of X and counting measure on the last $d - d_1$ coordinates.

Let $\mathscr{B}$ denote the collection of relatively compact Borel subsets of X. Let $\mathscr{A}$ denote the subcollection of $\mathscr{B}$ of sets A such that $|\partial A| = 0$.

For $x, y \in R^d$ set

$$x \cdot y = x^1 y^1 + \cdots + x^d y^d.$$

Then "$\cdot$" defines an inner product on R^d. Given any vector $v \in R^d$ of unit length, by applying the Gram-Schmidt Orthogonalization process to v, $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)$, we get an orthonormal basis $v_1 = v, v_2, \dots, v_d$ of R^d.

From now on v will denote an arbitrary but fixed vector in R^d of unit length. Whenever it can be done unambiguously, we will suppress the dependence on v of various quantities defined below.

Let

$$Z_v^{d-1} = \{n_2 v_2 + \cdots + n_d v_d \mid n_2, \dots, n_d \text{ are integers}\}$$

and

$$\Gamma_m = \Gamma_{m,v} = \{x_1 v_1 + x_2 v_2 + \cdots + x_d v_d \mid 0 \leqq x_k < m \text{ for } 2 \leqq k \leqq d\}.$$

Let $\mathscr{T}_v$ denote the collection of sets

$$T = T_{a_1,a_2} = T_{v,a_1,a_2} = \{x \in R^d \mid a_1 \leqq x \cdot v \leqq a_2\},$$

where a_1 and a_2 denote finite real numbers.

All measures considered below will be positive measures on the Borel sets in R^d. Furthermore, it will *always* be assumed that *these measures are supported by* X.

For $0 \leqq \lambda < \infty$, let $\mathscr{N}_{v,\lambda}$ denote the collection of measures ν supported by T for some $T \in \mathscr{T}_v$ (and also by X as remarked above) and such that

$$\lim_{m \to \infty} \frac{\nu(x + \Gamma_m)}{m^{d-1}} = \lambda \tag{2.1}$$

uniformly for $x \in R^d$.

For $0 \leqq \kappa < \infty$, let $\mathscr{M}_{v,\kappa}$ denote the collection of measures μ (supported by X) such that

$$\lim_{a_1, a_2 - a_1 \to \infty} \frac{\mu(T_{a_1,a_2})}{a_1 - a_2} = \kappa \tag{2.2}$$

and for each $0 < a < \infty$, $u \in U$, and $T \in \mathscr{T}_v$

$$\lim_{c \to \infty} \sum_{cv+T} |\mu(a \odot (n+u+\Delta)) - \mu(a \odot (n+\Delta))| = 0. \tag{2.3}$$

Here "$cv+T$" is used as an abbreviation for "$n \in Z^d \cap (cv+T)$." Similar abbreviations will be used below without further comment.

Let $\mathscr{M}^*_{v,\kappa}$ consist of the subcollection of $\mathscr{M}_{v,\kappa}$ of measures such that for $0 < a < \infty$, $T \in \mathscr{T}_v$, compact subset C of X, and $A \in \mathscr{B}$ with $|A| = 1$.

$$\lim_{c \to \infty} \sum_{cv+T} \sup_{x \in C} |\mu(a \odot (n+x+A)) - \mu(a \odot (n+\Delta))| = 0. \tag{2.4}$$

When the expression "$x \cdot v \to \infty$" is used below, it should be understood that $x \in X$.

THEOREM 1. *Let ν be in $\mathscr{N}_{v,\lambda}$. Let μ be in $\mathscr{M}_{v,\kappa}$ (or in $\mathscr{M}^*_{v,\kappa}$). Then for $A \in \mathscr{A}$ (or $A \in \mathscr{B}$)*

$$\lim_{x \cdot v \to \infty} (\nu * \mu)(x+A) = \lambda\kappa|A|. \tag{2.5}$$

As a first converse, suppose ν is supported by some $T \in \mathscr{T}_v$, $\mu(T) < \infty$ for all $T \in \mathscr{T}_v$, and, for some $0 < a < \infty$ and $0 < \alpha < \infty$,

$$\lim_{x \cdot v \to \infty} (\nu * \mu)(x + a \odot \Delta) = \alpha|a \odot \Delta|. \tag{2.6}$$

Then there exist $\lambda > 0$ and $\kappa > 0$ such that $\lambda\kappa = \alpha$, $\nu \in \mathscr{N}_{v,\lambda}$, and μ satisfies (2.2).

As a second converse suppose $0 < \kappa < \infty$ and, for all $0 < a < \infty$, $0 < \lambda < \infty$, and $\nu \in \mathscr{N}_{v,\lambda}$, that (2.6) *holds with $\alpha = \lambda\kappa$. Then $\mu \in \mathscr{M}_{v,\kappa}$.*

The apparent difference between (2.3) and (2.4) is minimized by the following result which is useful in proving the first part of Theorem 1.

LEMMA 1. *Let $\mu \in \mathscr{M}_{v,\kappa}$. Then for $0 < a < \infty$, $T \in \mathscr{T}_v$, and compact subset C of X*

$$\lim_{c \to \infty} \sum_{cv+T} \sup_{x \in C} |\mu(a \odot (n+x+\Delta)) - \mu(a \odot (n+\Delta))| = 0. \tag{2.7}$$

In Theorems 2 and 3, N denotes a "random positive measure" on the Borel sets in R^d (supported by X as usual).

THEOREM 2. *Let $0 \leqq \kappa < \infty$ and $\mu \in \mathscr{M}_{v,\kappa}$ (or $\mathscr{M}^*_{v,\kappa}$). Let $0 \leqq \lambda < \infty$ and let N be such that $EN \in \mathscr{N}_{v,\lambda}$ and*

$$\lim_{m \to \infty} E\left|\frac{N(x+\Gamma_m)}{m^{d-1}} - \lambda\right| = 0 \tag{2.8}$$

uniformly for $x \in R^d$. Then for $A \in \mathscr{A}$ (or $A \in \mathscr{B}$)

$$\lim_{x \cdot v \to \infty} E|(N * \mu)(x+A) - \lambda\kappa|A|\,| = 0. \tag{2.9}$$

Conversely, suppose $0<\kappa<\infty$, $0<a<\infty$, $0<\lambda<\infty$, μ *satisfies* (2.2), $EN\in\mathcal{N}_{v,\lambda}$, *and*

$$\lim_{x\cdot v\to\infty} E|(N*\mu)(x+a\odot\Delta)-\kappa\lambda|a\odot\Delta|\,| = 0. \tag{2.10}$$

Then (2.8) *holds.*

It follows from (2.9) that

$$\lim_{x\cdot v\to\infty} (N*\mu)(x+A) \overset{\text{Prob}}{=} \lambda\kappa|A|. \tag{2.11}$$

We next obtain analogous results when $\kappa=\infty$.

For $T\in\mathcal{T}_v$, we will say that T is *sufficiently large* if for some $m\geqq 1$

$$(x+\Gamma_m)\cap Z^d\cap T\neq\varnothing, \qquad x\in R^d. \tag{2.12}$$

THEOREM 3. *Let A be a Borel subset of X.*

(i) *Suppose that for sufficiently large* $T\in\mathcal{T}_v$ *and for all compact subsets C of X*

$$\lim_{x\cdot v\to\infty}\sum_T \inf_{y\in C}\mu(n+x+y+A) = \infty. \tag{2.13}$$

Then for $0<\lambda<\infty$ *and* $\nu\in\mathcal{N}_{v,\lambda}$

$$\lim_{x\cdot v\to\infty} (\nu*\mu)(x+A) = \infty. \tag{2.14}$$

(ii) *In order that for all* $0<\lambda<\infty$ *and* $\nu\in\mathcal{N}_{v,\lambda}$

$$(\nu*\mu)(x+A) = \infty, \qquad x\in X, \tag{2.15}$$

it is necessary and sufficient that for sufficiently large $T\in\mathcal{T}_v$ *and for all compact subsets C of X*

$$\sum_T \inf_{y\in C}\mu(n+y+A) = \infty. \tag{2.16}$$

THEOREM 4. *Let N be such that for some* λ, $0<\lambda<\infty$, $EN\in\mathcal{N}_{v,\lambda}$ *and* (2.8) *holds.*

(i) *If* (2.13) *holds for sufficiently large* $T\in\mathcal{T}_v$ *and for all compact subsets C of X, then*

$$\lim_{x\cdot v\to\infty} (N*\mu)(x+A) \overset{\text{P}}{=} \infty. \tag{2.17}$$

(ii) *If* (2.16) *holds for sufficiently large* $T\in\mathcal{T}_v$ *and all compact subsets C of X, then*

$$\mathrm{P}\{(N*\mu)(x+A) = \infty \text{ for } x\in X\} = 1. \tag{2.18}$$

3. **Proof of Theorem 1.** Set

$$Z_m^d = \{n\in Z^d \mid m^{-1}\odot n\in\Delta\}.$$

We first prove Lemma 1, starting off with

LEMMA 2. *Let μ denote a probability measure on X. Then for $m \geqq 1$ there is a $T_1 \in \mathscr{T}_v$ depending only on T and m such that*

$$\sum_{cv+T} \max_{k \in Z_m^d} \mu(m \odot n+k+\Delta) \leqq m^{-d_1}\mu(cv+T_1) + \sum_{cv+T_1} \max_{u \in U} |\mu(n+u+\Delta)-\mu(n+\Delta)|. \tag{3.1}$$

Proof of Lemma 2. For $n \in Z^d$ choose $b_n \in Z_m^d$ such that

$$\mu(m \odot n+b_n+\Delta) = \max_{k \in Z_m^d} \mu(m \odot n+k+\Delta).$$

Then for $k \in Z_m^d$

$$\mu(m \odot n+b_n+\Delta)-\mu(m \odot n+k+\Delta) \leqq \sum_{z \in Z_m^d} \max_{u \in U} |\mu(m \odot n+z+u+\Delta)-\mu(m \odot n+z+\Delta)|.$$

Consequently

$$\begin{aligned}\sum_{cv+T} \mu(m \odot n+b_n+\Delta) &\leqq \sum_{cv+T} \mu(m \odot n+k+\Delta) \\ &\quad + \sum_{cv+T} \sum_{z \in Z_m^d} \max_{u \in U} |\mu(m \odot n+z+u+\Delta)-\mu(m \odot n+z+\Delta)| \\ &\leqq \sum_{cv+T} \mu(m \odot n+k+\Delta) + \sum_{cv+T_2} \max_{u \in U} |\mu(n+u+\Delta)-\mu(n+\Delta)|\end{aligned}$$

for suitable $T_2 \in \mathscr{T}_v$ depending only on T and m. Choose $T_1 \supseteq T_2$ such that $m \odot n+k+\Delta \subseteq T_1$ for $n \in cv+T$ and $k \in Z_m^d$. Then Lemma 2 follows by summing over $k \in Z_m^d$ and dividing by m^d.

Proof of Lemma 1. Without loss of generality we can assume that $a=1$ and $C=\Delta$. Then (2.7) becomes

$$\lim_{c \to \infty} \sum_{cv+T} \sup_{x \in \Delta} |\mu(n+x+\Delta)-\mu(n+\Delta)| = 0. \tag{3.2}$$

Observe that

$$\begin{aligned}&\limsup_{c \to \infty} \sum_{cv+T} \max_{k \in Z_m^d} |\mu(n+m^{-1} \odot k+\Delta)-\mu(n+\Delta)| \\ &= \limsup_{c \to \infty} \sum_{cv+T} \max_{k \in Z_m^d} |\mu(m^{-1} \odot (m \odot n+k+m \odot \Delta))-\mu(m^{-1} \odot (m \odot n+m \odot \Delta))| \\ &\leqq \limsup_{c \to \infty} \sum_{cv+T} \sum_{k \in Z_m^d} \max_{v \in U} |\mu(m^{-1} \odot (m \odot n+k+u+m \odot \Delta)) \\ &\qquad -\mu(m^{-1} \odot (m \odot n+k+m \odot \Delta))| \\ &\leqq \limsup_{c \to \infty} \sum_{cv+T_1} \max_{u \in U} |\mu(m^{-1} \odot (n+u+m \odot \Delta))-\mu(m^{-1} \odot (n+m \odot \Delta))|\end{aligned}$$

for a suitable $T_1 \in \mathscr{T}_v$. It now follows from (2.3) that

$$\lim_{c \to \infty} \sum_{cv+T} \max_{k \in Z_m^d} |\mu(n+m^{-1} \odot k+\Delta)-\mu(n+\Delta)| = 0. \tag{3.3}$$

By Lemma 2

$$\sum_{cv+T} \max_{k \in Z_m^d} \mu(n+m^{-1} \odot k+m^{-1} \odot \Delta)$$
$$= \sum_{cv+T} \max_{k \in Z_m^d} \mu\left(\frac{1}{m} \odot (m \odot n+k+\Delta)\right)$$
$$\leqq m^{d_1}\mu(cv+T_1) + \sum_{cv+T_1} \max_{u \in U} \left|\mu\left(\frac{1}{m} \odot (n+u+\Delta)\right) - \mu\left(\frac{1}{m} \odot (n+\Delta)\right)\right|$$

for some T_1 depending on T, but which, as is easily seen, can be chosen independently of m. It now follows from (2.2) and (2.3) that for some constant β_1, $0<\beta_1<\infty$, depending on T

$$\limsup_{c\to\infty} \sum_{cv+T} \max_{k \in Z_m^d} \mu(n+m^{-1} \odot k+m^{-1} \odot \Delta) \leqq \beta_1 m^{-d_1}. \tag{3.4}$$

With no loss of generality, we can assume that $d_1 \geqq 1$. Let W denote the set of 2^{d_1} points in Z^d whose coordinates are all zero or one, the last $d-d_1$ of them all being zero. Choose $x \in \Delta$. Choose $k=(k^1, \ldots, k^d) \in Z_m^d$ such that

$$m^{-1}k^i \leqq x^i < m^{-1}(k^i+1), \qquad 1 \leqq i \leqq d.$$

Then

$$|\mu(n+x+\Delta)-\mu(n+m \odot k+\Delta)|$$
$$\leqq 2^{d_1}(m+1)^{d_1-1} \sum_{w\in W} \max_{k \in Z_m^d} \mu(n+w+m^{-1} \odot k+m^{-1} \odot \Delta).$$

Thus by (2.4), there is a constant β, $0<\beta<\infty$, depending only on T and such that

$$\limsup_{c\to\infty} \sum_{cv+T} \sup_{x\in X} \min_{k\in Z_m^d} |\mu(n+x+\Delta)-\mu(n+m^{-1} \odot k+\Delta)| \leqq \beta m^{-1}. \tag{3.5}$$

Choose $\varepsilon>0$. Choose m such that

$$\beta m^{-1} \leqq \varepsilon/2. \tag{3.6}$$

By (3.3), (3.5), and (3.6), there is a c_0 such that for $c \geqq c_0$

$$\sum_{cv+T} \max_{k\in Z_m^d} |\mu(n+m^{-1} \odot k+\Delta)-\mu(n+\Delta)| \leqq \frac{\varepsilon}{2}$$

and

$$\sum_{cv+T} \sup_{x\in\Delta} \min_{k\in Z_m^d} |\mu(n+x+\Delta)-\mu(n+m^{-1} \odot k+\Delta)| \leqq \frac{\varepsilon}{2}.$$

Therefore for $c \geqq c_0$

$$\sum_{cv+T} \sup_{x\in\Delta} |\mu(n+x+\Delta)-\mu(n+\Delta)| \leqq \varepsilon.$$

Since ε can be made arbitrarily small, this completes the proof of (3.2) and hence also of Lemma 1.

Proof of first part of Theorem 1. With no loss of generality, we can assume that $\lambda=\kappa=1$. Choose $\nu \in \mathcal{N}_{v,1}$ and $\mu \in \mathcal{M}_{v,1}$. Let $T_1 \in \mathcal{T}_v$ be such that ν is supported on T_1.

In proving that (2.5) holds for $A \in \mathcal{A}$, it suffices to consider the case $A=\Delta$. If $\mu \in \mathcal{M}^*_{v,1}$, then to prove that (2.5) holds for $A \in \mathcal{B}$, it suffices to consider A such that $|A|=1$. In either case, by Lemma 1, it suffices to consider $A \in \mathcal{B}$ such that $|A|=1$ and (2.4) holds for $a=1$, $T \in \mathcal{T}_v$, and C a compact subset of X. From now on we let A be such a set.

Choose $0 \leqq \varepsilon \leqq 1$. Choose $m_1>0$ such that for $m \geqq m_1$

$$1-\varepsilon \leqq m^{1-d}\nu(x-\Gamma_m) \leqq 1+\varepsilon, \qquad x \in R^d. \tag{3.7}$$

Consider $T \in \mathcal{T}_v$ of the form

$$T = \{x \in R^d \mid 0 \leqq x\cdot v \leqq a\}.$$

By (2.2) there exist $a_1>0$ and $c_1>0$ such that if $a \geqq a_1$ and $c \geqq c_1$ then

$$(1-\varepsilon)a \leqq \mu(cv+T) \leqq (1+\varepsilon)a. \tag{3.8}$$

Let $b'_m(T)$ denote the minimum (over $-\infty<c<\infty$ and $x \in R^d$) of the number of points $n \in Z^d$ such that

$$n+\Delta \subseteq (cv+T) \cap (x+\Gamma_m).$$

Let $b''_m(T)$ denote the maximum (over $-\infty<c<\infty$ and $x \in R^d$) of the number of points $n \in Z^d$ such that

$$(n+\Delta) \cap (cv+T) \cap (x+\Gamma_m) \neq \varnothing.$$

Then clearly

$$b'_m(T) \leqq am^{d-1} \leqq b''_m(T).$$

There exist $m_2 \geqq m_1$ and $a_2 \geqq a_1$ such that if $m \geqq m_2$ and $a \geqq a_2$, then

$$(1-\varepsilon)am^{d-1} \leqq b'_m(T) \tag{3.9}$$

and

$$(1+\varepsilon)am^{d-1} \geqq b''_m(T). \tag{3.10}$$

Assume now that a and m are fixed with $a \geqq a_2$ and $m \geqq m_2$. By (2.4) there is a $c_2 \geqq c_1$ such that whenever $c \geqq c_2$ and the points $n_z \in Z^d$, $z \in Z_v^{d-1}$ are such that

$$(n_z+\Delta) \cap (cv+T) \cap (mz+\Gamma_m) \neq \varnothing, \tag{3.11}$$

then

$$\sum_{z \in Z_v^{d-1}} \sup_{y \in Y_z} |\mu(y+A)-\mu(n_z+\Delta)| \leqq \frac{\varepsilon}{2m^{d-1}}, \tag{3.12}$$

where

$$Y_z = X \cap (mz+\Gamma_m) \cap (cv-T_1).$$

Choose $x \in X$ such that $x \cdot v = c \geqq c_2$. Then

$$(3.13)\quad (\nu * \mu)(x+A) = \int_{X \cap (cv - T_1)} \nu(x-dy)\mu(y+A) = \sum_{z \in Z_v^{d-1}} \int_{Y_z} \nu(x-dy)\mu(y+A).$$

Thus by (3.7), (3.12), and (3.13), whenever $n_z \in Z^d$ satisfies (3.11), then

$$(3.14)\quad \left| (\nu * \mu)(x+A) - \sum_{z \in Z_v^{d-1}} \nu(x - mz - \Gamma_m)\mu(n_z + \Delta) \right| \leqq \varepsilon(1+\varepsilon)/2 \leqq \varepsilon.$$

We can get $b'_m(T)$ different choices of the sequence n_z, $z \in Z_v^{d-1}$, such that all the n_z's involved are distinct and, for each such n_z,

$$n_z + \Delta \subseteq cv + T.$$

Summing (3.14) and using (3.7), we get that

$$(3.15)\quad b'_m(T)(\nu * \mu)(x+A) \leqq (1+\varepsilon)m^{d-1}\mu(cv+T) + \varepsilon b'_m(T)$$

and hence by (3.8) and (3.9) that

$$(3.16)\quad (\nu * \mu)(x+A) \leqq (1+\varepsilon)^2/(1-\varepsilon) + \varepsilon.$$

Similarly we can get $b''_m(T)$ different choices of the sequence n_z, $z \in Z_v^{d-1}$, such that every n_z satisfying

$$(n_z + \Delta) \cap (cv + T) \neq \varnothing$$

is included in at least one such sequence. Summing on (3.14) and using (3.7), we get that

$$(3.17)\quad b''_m(T)(\nu * \mu)(x+A) \geqq (1-\varepsilon)m^{d-1}\mu(cv+T) - \varepsilon b''_m(T)$$

and hence by (3.8) and (3.10) that

$$(3.18)\quad (\nu * \mu)(x+A) \geqq (1-\varepsilon)^2/(1+\varepsilon) - \varepsilon.$$

Since ε can be made arbitrarily small, (2.5) follows from (3.16) and (3.18).

Proof of first converse of Theorem 1. We can assume that $a = \alpha = 1$. Let ν be supported by $T_{a_0, b_0} \in \mathscr{T}_v$, let $\mu(T) < \infty$ for $T \in \mathscr{T}_v$, and suppose that

$$(3.19)\quad \lim_{x \cdot v \to \infty} (\nu * \mu)(x + \Delta) = 1.$$

It follows easily from (3.19) that if a and $b-a$ are sufficiently large, then $\mu(T_{a,b}) > 0$. It also follows easily from (3.19) that

$$(3.20)\quad \frac{(\nu * \mu)(x + \Gamma_m \cap T_{a,b})}{m^{d-1}(b-a)} \to 1$$

as $x \cdot v = 0$, $m \to \infty$, $a \to \infty$, and $b - a \to \infty$.

In order to prove the desired converse, we need only show that
for every $0 \leqq \varepsilon \leqq \frac{1}{2}$ *there exist* a_1 *and* d_1 *such that if* $a \geqq a_1$ *and* $b-a \geqq d_1$, *then*

$$\limsup_{m\to\infty} \sup_{x\in R^d} \frac{\nu(x+\Gamma_m)}{m^{d-1}} \leqq \frac{(1+\varepsilon)(b-a)}{\mu(T_{a,b})} \tag{3.21}$$

and

$$\liminf_{m\to\infty} \inf_{x\in R^d} \frac{\nu(x+\Gamma_m)}{m^{d-1}} \geqq \frac{(1-\varepsilon)(b-a)}{\mu(T_{a,b})}. \tag{3.22}$$

Choose $0 < \varepsilon \leqq \frac{1}{2}$. It follows from (3.20) that there exist positive numbers m_2, a_2, and d_2 such that

$$1-\frac{\varepsilon}{8} < \frac{(\nu * \mu)(x+\Gamma_m \cap T_{a,b})}{m^{d-1}(b-a)} < 1+\frac{\varepsilon}{8} \tag{3.23}$$

for $x\cdot v=0$, $m \geqq m_2$, $a \geqq a_2$, and $b-a \geqq d_2$. Choose $a_1 \geqq a_2 - a_0$ and $d_1 \geqq d_2 + b_0 - a_0$ such that $b_0 - a_0 \leqq \varepsilon d_1/8$ and $\mu(T_{a,b}) > 0$, $a \geqq a_1$ and $b-a \geqq d_1$.

We shall show that a_1 and d_1 are the numbers desired for (3.21) and (3.22). Choose a and b such that $a \geqq a_1$ and $b-a \geqq d_1$. There is an m_1 such that

$$\mu(T_{a,b} \cap \Gamma_{m_1}) \leqq (1-\varepsilon/8)\mu(T_{a,b}).$$

Suppose (3.21) does not hold for this choice of a and b. Then we can find $m \geqq m_2$ and $x_0 \in R^d$ such that $x_0 \cdot v = 0$ and

$$\frac{\nu(x_0-y+\Gamma_m)}{m^{d-1}} \geqq \left(1+\frac{\varepsilon}{2}\right)\frac{b-a}{\mu(T_{a,b})}, \qquad y \in \Gamma_{m_1}.$$

Consequently

$$\begin{aligned}(\nu * \mu)(x_0+\Gamma_m \cap T_{a+a_0,b+b_0}) &= \int \nu(x_0-y-\Gamma_m \cap T_{a+a_0,b+b_0})\mu(dy)\\ &\geqq \int_{T_{a,b}} \nu(x_0-y+\Gamma_m \cap T_{a+a_0,b+b_0})\mu(dy)\\ &= \int_{T_{a,b}} \nu(x_0-y+\Gamma_m)\mu(dy)\\ &\geqq \int_{T_{a,b}\cap\Gamma_{m_1}} \nu(x_0-y+\Gamma_m)\mu(dy)\\ &\geqq \left(1+\frac{\varepsilon}{2}\right)\frac{(b-a)\mu(T_{a,b}\cap\Gamma_{m_1})}{\mu(T_{a,b})}\\ &\geqq (1+\varepsilon/2)(1-\varepsilon/8)(1-\varepsilon/8)m^{d-1}(b+b_0-a-a_0)\\ &\geqq (1+\varepsilon/8)m^{d-1}(b+b_0-a-a_0),\end{aligned}$$

which contradicts (3.23). Thus we have verified that (3.21) holds. Suppose (3.22) does not hold. Then we can find $x_0 \in R^d$ and m such that $x_0\cdot v=0$, $m \geqq m_2$,

$$\frac{\nu(x_0-y+\Gamma_m)}{m^{d-1}} \leqq \frac{(1-\varepsilon/2)(b-a)}{\mu(T_{a,b})}, \qquad y \in \Gamma_{m_1},$$

and

$$\frac{\nu(x_0-y+\Gamma_m)}{m^{d-1}} \leqq \frac{(1+3\varepsilon/2)(b-a)}{\mu(T_{a,b})}, \qquad y \in R^d.$$

Then

$$
\begin{aligned}
(\nu * \mu)(x_0+\Gamma_m \cap T_{a+b_0,b+a_0}) &= \int \nu(x_0-y+\Gamma_m \cap T_{a+b_0,b+a_0})\mu(dy) \\
&\leqq \int_{T_{a,b}} \nu(x_0-y+\Gamma_m)\mu(dy) \\
&\leqq \left(1-\frac{\varepsilon}{2}\right)(b-a)m^{d-1}+\frac{\varepsilon}{8}\left(1+\frac{3\varepsilon}{2}\right)(b-a)m^{d-1} \\
&\leqq \left(1-\frac{\varepsilon}{4}\right)(b-a)m^{d-1} \leqq \left(1-\frac{\varepsilon}{8}\right)(b+a_0-a-b_0)m^{d-1},
\end{aligned}
$$

which contradicts (3.23). This completes the proof of (3.22) and hence of the first converse of Theorem 1.

Proof of second converse of Theorem 1. It follows easily from the first converse of Theorem 1 and the assumptions of the second converse that μ satisfies (2.2). We can assume that $\kappa=1$. By assumption we have that if $0<a<\infty$, $\nu \in \mathcal{N}_{v,1}$ and ν is supported by $a \odot Z^d$, then (1.6) holds. We need to show that, for $u \in U$ and $T \in \mathcal{T}_v$,

$$
\lim_{c\to\infty} \sum_{cv+T} |\mu(a \odot (n+u+\Delta))-\mu(a \odot (n+\Delta))| = 0. \tag{2.3}
$$

With no loss of generality, we can assume that $a=1$. Then ν is supported by Z^d and (2.3) becomes

$$
\lim_{c\to\infty} \sum_{cv+T} |\mu(n+u+\Delta)-\mu(n+\Delta)| = 0. \tag{3.24}
$$

The general case now reduces to the lattice case $d_1=0$ and $X=Z^d$.

Assume now that $d_1=0$ and $X=Z^d$. For $n \in Z^d$ set $\nu(n)=\nu(\{n\})$ and $\mu(n)=\mu(\{n\})$. We have that

$$
\lim_{a,b-a\to\infty} \frac{1}{b-a} \sum_{a\leqq n\cdot v\leqq b} \mu(n) = 1. \tag{3.25}
$$

We also have that whenever $\nu \in \mathcal{N}_{v,1}$, then

$$
\lim_{k\cdot v\to\infty} \sum \nu(-n)\mu(k+n) = 1. \tag{3.26}
$$

We want to prove that for $u \in U$ and $T \in \mathcal{T}_v$

$$
\lim_{c\to\infty} \sum_{cv+T} |\mu(n+\mu)-\mu(n)| = 0. \tag{3.27}
$$

It follows from (3.26) that

$$
\lim_{k\cdot v\to\infty} \mu(k) = 0 \tag{3.28}
$$

and that

$$
\lim_{k\cdot v\to\infty} \sum \nu(-n)(\mu(k+u+n)-\mu(k+n)) = 0. \tag{3.29}
$$

In order to prove that (2.36) holds, we will suppose that for some $u_0 \in U$ and $T_0 = T_{a_0,b_0} \in \mathcal{T}_v$

$$\limsup_{c\to\infty} \sum_{cv+T_0} |\mu(n+u_0)-\mu(n)| > 8\alpha > 0, \tag{3.30}$$

but (3.25) and (3.28) hold. We will then find a $\nu \in \mathcal{N}_{v,1}$ such that (3.29) does not hold.

If we modify T_0 by making it bigger, then (3.30) still remains valid. Thus we can assume that T_0 is large enough so that there exists m_0 such that for $m \geqq m_0$ and $x \in R^d$ the number of points in $Z^d \cap (x+T_0 \cap \Gamma_m)$ is bounded above by

$$2m^{d-1}(b_0-a_0),$$

or in other words

$$\sum_{Z^d\cap(x+T_0\cap\Gamma_m)} 1 \leqq 2m^{d-1}(b_0-a_0). \tag{3.31}$$

Also by (3.25) we can find $T_1 = T_{a_1,b_1} \in \mathcal{T}_v$ such that $T_1 \supseteq T_0$ and, for some c_1,

$$\mu(cv+T_1) \leqq 2(b_1-a_1), \qquad c \geqq c_1. \tag{3.32}$$

An elementary argument shows that T_1 can be made large enough so that there exists $m_1 \geqq m_0$ such that for $m \geqq m_1$, $x \in R^d$, $u \in U$, and $-\infty < \beta < 0$, if

$$\mu(n+u)-\mu(n) \leqq \beta, \qquad n \in x+T_1 \cap \Gamma_m,$$

then

$$\mu(x+T_1 \cap \Gamma_m) \geqq 2\alpha^{-1}(-\beta)(b_0-a_0)m^{d-1}(b_1-a_1). \tag{3.33}$$

Set $C = \{c \mid -\infty < c < \infty\}$. Let C_1 denote the subset of $c \in C$ such that

$$\sum_{cv+T_0} |\mu(n+u_0)-\mu(n)| \geqq 8\alpha.$$

Set

$$P = \{n \in Z^d \mid \mu(n+u_0)-\mu(n) > 0\}.$$

Let C_2 denote the subset of C_1 on which

$$\sum_{P\cap(cv+T_0)} (\mu(n+u_0)-\mu(n)) \geqq 4\alpha. \tag{3.34}$$

With no loss of generality we can assume that C_2 is unbounded from above.

Choose $m \geqq m_1$. For $c \in C$ and $z \in Z_v^{d-1}$, let $n_z(c)$ be in $cv+mz+\Gamma_m \cap T_1$ and such that $\mu(n_z(c)+u_0)-\mu(n_z(c))$ maximizes $\mu(n+u_0)-\mu(n)$ over all $n \in cv+mz+\Gamma_m \cap T_1$. Let

$$[cv] = ([cv^1], \ldots, [cv^d]),$$

where $[cv^k]$ denotes the greatest integer in cv^k. Set $S_c = \{n_z(c) \mid z \in Z_v^{d-1}\}$ and let ν_c be the measure supported on $[cv]-S_c$ and defined by

$$\nu_c(\{[cv]-n_z(c)\}) = m^{d-1}. \tag{3.35}$$

Then $\nu_c \in \mathcal{N}_{v,1}$. Set $\nu_c(n)=\nu_c(\{n\})$. For $c \in C$

(3.36) $\sum \nu_c(-n)(\mu(n+[cv]+u_0)-\mu(n+[cv])) = m^{d-1} \sum_{n\in S_c} (\mu(n+u_0)-\mu(n)).$

For $n=n_z(c) \in P \cap S_c$ we have by (3.31) that

$$m^{d-1}(\mu(n+u_0)-\mu(n)) \geqq \frac{1}{2(b_0-a_0)} \sum_{(cv+mz+\Gamma_m\cap T_0)\cap P} (\mu(n+u_0)-\mu(n)).$$

Thus by (3.34) for $c \in C_2$

$$m^{d-1} \sum_{P\cap S_c} (\mu(n+u_0)-\mu(n)) \geqq \frac{2\alpha}{b_0-a_0}. \tag{3.37}$$

For $n=n_z(c) \in (Z^d-P) \cap S_c$ (i.e. in S_c and the complement of P), we have by (3.33) that

$$m^{d-1}(\mu(n+u_0)-\mu(n)) \geqq -\frac{\alpha\mu(cv+mz+\Gamma_m \cap T_1)}{2(b_0-a_0)(b_1-a_1)}.$$

Therefore

$$m^{d-1} \sum_{(Z^d-P)\cap S_c} (\mu(n+u_0)-\mu(n)) \geqq -\frac{\alpha\mu(cv+T_1)}{2(b_0-a_0)(b_1-a_1)} \geqq -\frac{\alpha}{b_0-a_0} \tag{3.38}$$

by (3.32). By (3.36), (3.37), and (3.38), we have that for $c \in C_2$

$$\sum \nu_c(-n)(\mu(n+[cv]+u_0)-\mu(n+[cv])) \geqq \frac{\alpha}{b_0-a_0}. \tag{3.39}$$

It is also clear that there is a set $T_2 \in \mathcal{T}_v$ such that ν_c is supported by T_2 for $c \in C_2$.

For $n \in Z^d$ set

$$|n| = ((n^1)^2+\cdots+(n^d)^2)^{1/2}.$$

We next define an increasing sequence of points $c_1, c_2, \ldots$ in C_2. Choose $c_1 \in C_2$ and set $a_0=0$. Once a_{i-1} and t_i are chosen, choose $a_i \geqq a_{i-1}+i$ such that

$$m^{d-1} \sum_{-T_3\cap\{|n|\geqq a_i\}} |\mu(n+[c_iv]+u_0)-\mu(n+[c_iv])| \leqq \frac{\alpha}{8(b_0-a_0)}. \tag{3.40}$$

Once c_i and a_i are chosen, choose $c_{i+1} \in C_2$ such that $c_{i+1} \geqq c_i+1$ and

$$m^{d-1} \sum_{-T_3\cap\{|n|<a_i\}} |\mu(n+[c_{i+1}v]+u_0)-\mu(n+[c_{i+1}v])| \leqq \frac{\alpha}{8(b_0-a_0)}. \tag{3.41}$$

This is possible by (3.28). Clearly

$$\lim_{i\to\infty} c_i = \infty. \tag{3.42}$$

Define ν by $\nu(\{n\})=\nu_{c_i}(\{n\})$, $a_{i-1}\leqq|n|<a_i$, and set $\nu(n)=\nu(\{n\})$. Then $\nu\in\mathscr{N}_{v,1}$ and for $i\geqq 1$,

$$\begin{aligned}\sum \nu(-n)(\mu(n+[c_iv]+u_0)-\mu(n+[c_iv]))\\ &\geqq \sum \nu_{c_i}(-n)(\mu(n+[c_iv]+u_0)-\mu(n+[c_iv]))\\ &\quad -2m^{d-1}\sum_{-T_3\cap\{|n|<a_i\}}|\mu(n+[c_iv]+u_0)-\mu(n+[c_iv])|\\ &\quad -2m^{d-1}\sum_{-T_3\cap\{|n|\geqq a_i\}}|\mu(n+[c_iv]+u_0)-\mu(n+[c_iv])|\\ &\geqq (\alpha/(b_0-a_0))(1-\tfrac{1}{4}-\tfrac{1}{4}) \geqq \alpha/2(b_0-a_0)\end{aligned}$$

by (3.39), (3.40), and (3.41).

In summary, $\nu\in\mathscr{N}_{v,1}$, $c_i\to\infty$, and

$$\sum \nu(-n)(\mu(n+[c_iv]+u_0)-\mu(n+[c_iv])) \geqq \frac{\alpha}{2(b_0-a_0)}. \tag{3.43}$$

This contradicts (3.29) and completes the proof of the theorem.

4. Proof of Theorems 2 and 3. Proof of first part of Theorem 2. It suffices to consider $\kappa=\lambda=1$ and $A\in\mathscr{B}$ such that $|A|=1$ and for $T\in\mathscr{T}_v$ and compact subset C of X

$$\lim_{c\to\infty}\sum_{cv+T}\sup_{y\in C}|\mu(n+y+A)-\mu(n+\Delta)| = 0. \tag{4.1}$$

Set $\nu=EN$. Choose $0\leqq\varepsilon\leqq 1$ and recall the definitions used in the first part of Theorem 1, starting with (3.7). By (2.8) we can also assume that

$$E|N(x-\Gamma_m)-\nu(x-\Gamma_m)| \leqq \varepsilon m^{d-1}, \qquad x\in R^d. \tag{4.2}$$

Now (3.4) can be replaced by

$$E\left|(N*\mu)(x+A)-\sum_{z\in Z_v^{d-1}}N(x-mz-\Gamma_m)\mu(n_z+\Delta)\right| \leqq \varepsilon. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned}&E\left|(N*\mu)(x+A)-\sum_{z\in Z_v^{d-1}}\nu(x-mz-\Gamma_m)\mu(n_z+\Delta)\right|\\ &\qquad\leqq \varepsilon+\varepsilon m^{d-1}\sum_{z\in Z_v^{d-1}}\mu(n_z+\Delta).\end{aligned} \tag{4.4}$$

Let S_1 denote the sum of

$$\sum_{z\in Z_v^{d-1}}\nu(x-mz-\Gamma_m)\mu(n_z+\Delta)$$

over $b_m''(T)$ different choices of the sequence n_z, $z\in Z_v^{d-1}$, satisfying (3.11) and such that every n_z satisfying

$$(n_z+\Delta)\cap(cv+T) \neq \varnothing$$

is included in at least one such sequence. Let S_2 denote the sum of

$$\sum_{z\in Z_v^{d-1}} \mu(n_z+\Delta)$$

over the same choice of sequences. It follows from (2.2) and (4.1) that, by changing slightly the construction used in the proof of the first part of Theorem 1, we can assume that

$$(1-\varepsilon)a \leqq S_2 \leqq (1+\varepsilon)a. \tag{4.5}$$

Summing (4.4) on the $b_m''(T)$ different sequences and using (4.5), we get that

$$E|b_m''(T)(N*\mu)(x+A)-S_1| \leqq \varepsilon b_m''(T)+\varepsilon(1+\varepsilon)m^{d-1}a.$$

Division by $b_m''(T)$ yields

$$E|(N*\mu)(x+A)-S_1/b_m''(T)| \leqq 3\varepsilon. \tag{4.6}$$

By (3.7) and (4.5)

$$(1-\varepsilon)^2am^{d-1} \leqq S_1 \leqq (1+\varepsilon)^2am^{d-1}.$$

Consequently

$$(1-\varepsilon)^2/(1+\varepsilon) \leqq S_1/b_m''(T) \leqq (1+\varepsilon)^2. \tag{4.7}$$

Since ε can be made arbitrarily small, (4.6) and (4.7) yield the conclusion of the first part of Theorem 2.

Proof of first converse of Theorem 2. With no loss of generality, we can assume that $\kappa=\lambda=a=1$. Then (2.9) becomes

$$\lim_{x\cdot v\to\infty} E|(N*\mu)(x+\Delta)-1| = 0. \tag{4.8}$$

It follows easily from this that

$$E\left|\frac{(N*\mu)(x+\Gamma_m\cap T_{a,b})}{m^{d-1}(b-a)}-1\right|\to 0 \tag{4.9}$$

as $x\cdot v=0$, $m\to\infty$, $a\to\infty$, and $b-a\to\infty$.

Let ν be supported by T_{a_0,b_0}. As in the proof of the first converse of Theorem 1, we get that

$$\int_{T_{a,b}} N(x-y-\Gamma_m)\mu(dy) \leqq (N*\mu)(x+\Gamma_m\cap T_{a+a_0,b+b_0}) \tag{4.10}$$

and

$$(N*\mu)(x+\Gamma_m\cap T_{a+b_0,b+a_0}) \leqq \int_{T_{a,b}} N(x-y+\Gamma_m)\mu(dy). \tag{4.11}$$

For $-\infty<x<\infty$ let $x^+=x$ and $x^-=0$ or $x^+=0$ and $x^-=x$ according as $x\geqq 0$ or $x<0$. In order to obtain (2.8) we need only prove that

$$\lim_{m\to\infty} E[(N(x+\Gamma_m)/m^{d-1}-1)^+] = 0 \tag{4.12}$$

and

$$\lim_{m\to\infty} E[(N(x+\Gamma_m)/m^{d-1}-1)^-] = 0, \tag{4.13}$$

both limits holding uniformly for $x \in R^d$.

We first prove (4.12), supposing it is false and obtaining a contradiction. If it is false, we can choose $0<\varepsilon<1$, m, m_1, m_2, a, b, and $x_0 \in R^d$ such that

$$N(x_0-y+\Gamma_m) \geqq N(x_0+\Gamma_{m_1}), \qquad y \in \Gamma_{m_2}, \tag{4.14}$$

$$E[((1-\varepsilon)N(x_0+\Gamma_{m_1})/m^{d-1}-1)^+] \geqq \varepsilon, \tag{4.15}$$

$$\mu(T_{a,b} \cap \Gamma_{m_2}) \geqq (1-\varepsilon)(b-a), \tag{4.16}$$

and

$$E\left[\left(\frac{(N*\mu)(x_0+\Gamma_m \cap T_{a+a_0,b+b_0})}{m^{d-1}(b-a)}-1\right)^+\right] < \varepsilon. \tag{4.17}$$

By (4.10), (4.14), and (4.16)

$$\begin{aligned}(N*\mu)(x_0+\Gamma_m \cap T_{a+a_0,b+b_0}) &\geqq \int_{T_{a,b},\cap\Gamma_{m_2}} N(x_0-y+\Gamma_m)\mu(dy)\\ &\geqq (1-\varepsilon)N(x_0+\Gamma_{m_1})(b-a).\end{aligned}$$

Thus by (4.15)

$$E\left[\left(\frac{(N*\mu)(x_0+\Gamma_m \cap T_{a+a_0,b+b_0})}{m^{d-1}(b-a)}-1\right)^+\right] \geqq E\left[\left(\frac{(1-\varepsilon)N(x_0+\Gamma_m)}{m^{d-1}}-1\right)^+\right] \geqq \varepsilon,$$

which contradicts (4.17). This completes the proof of (4.12).

We next prove (4.13), again supposing it is false and obtaining a contradiction. If it is false we can choose $0<\varepsilon<1$, m, m_1, m_2, a, b, and $x_0 \in R^d$ such that

$$N(x_0-y+\Gamma_m) \leqq N(x_0+\Gamma_{m_1}), \qquad y \in \Gamma_{m_2}, \tag{4.18}$$

$$E\left[\left(\frac{(1+\varepsilon)N(x_0+\Gamma_{m_1})}{m^{d-1}}-1\right)^-\right] \geqq 2\varepsilon, \tag{4.19}$$

$$\mu(T_{a,b}) \leqq (1+\varepsilon)(b-a), \tag{4.20}$$

$$\mu(T_{a,b})-\mu(T_{a,b} \cap \Gamma_{m_2}) \leqq \varepsilon(b-a)/2, \tag{4.21}$$

$$EN(x+\Gamma_{m_1}) \leqq 2m^d, \qquad x \in R^d, \tag{4.22}$$

and

$$E\left[\left(\frac{(N*\mu)(x_0+\Gamma_m \cap T_{a+b_0,b+a_0})}{m^{d-1}(b-a)}-1\right)^-\right] < \varepsilon. \tag{4.23}$$

By (4.11), (4.18), and (4.20)

$$(N*\mu)(x_0+\Gamma_m\cap T_{a+b_0,b+a_0}) \leqq (1+\varepsilon)N(x_0+\Gamma_{m_1})(b-a) + \int_{T_{a,b}\cap\Gamma^c_{m_2}} N(x_0-y+\Gamma_m)\mu(dy)$$

and hence

$$\left(\frac{(N*\mu)(x_0+\Gamma_m\cap T_{a+b_0,b+a_0})}{m^{d-1}(b-a)}-1\right)^- \geqq \left(\frac{(1+\varepsilon)N(x_0+\Gamma_{m_1})}{m^{d-1}}-1\right)^- - \frac{1}{m^{d-1}(b-a)}\int_{T_{a,b}\cap\Gamma^c_{m_2}} N(x_0-y-\Gamma_m)\mu(dy).$$

Thus by (4.19), (4.21), and (4.22), we get that

$$E\left[\left(\frac{(N*\mu)(x_0+\Gamma_m\cap T_{a+b_0,b+a_0})}{m^{d-1}(b-a)}-1\right)^-\right] > 2\varepsilon-\varepsilon \geqq \varepsilon,$$

which contradicts (4.23). This completes the proof of Theorem 2.

Proof of Theorem 3(i). Let $T\in\mathscr{T}_v$ be sufficiently large and let $m\geqq 1$ be such that (2.12) holds. Then there is a compact subset C of X such that

$$n+C \supseteq (mz+\Gamma_m)\cap T\cap X, \qquad z\in Z_v^{d-1} \text{ and } n\in(mz+\Gamma_m)\cap T\cap Z^d.$$

Let $0<\lambda<\infty$ and $\nu\in\mathscr{N}_{v,\lambda}$. We can assume that m is such that

$$\nu(-(mz+\Gamma_m)) \geqq \lambda m^{d-1}/2, \qquad z\in Z_v^{d-1},$$

and T is such that ν is supported by $-T$. Let A be a Borel subset of X. Then for $x\in X$

$$(\nu*\mu)(x+A) = \sum_{z\in Z_v^{d-1}}\int_{(mz+\Gamma_m)\cap T}\nu(-dy)\mu(x+y+A)$$
$$\geqq \frac{\lambda m^{d-1}}{2}\sum_{z\in Z_v^{d-1}}\inf_{y\in(mz+\Gamma_m)\cap T\cap X}\mu(x+y+A).$$

Suppose (2.13) holds for this A. Then

$$\lim_{x\cdot v\to\infty}\sum_{z\in Z_v^{d-1}}\sum_{(mz+\Gamma_m)\cap T}\inf_{y\in C}\mu(n+x+y+A) = \infty.$$

Consequently

$$\lim_{x\cdot v\to\infty}\sum_{z\in Z_v^{d-1}}\max_{(mz+\Gamma_m)\cap T}\inf_{y\in C}\mu(n+x+y+A) = \infty$$

and hence

$$\lim_{x\cdot v\to\infty}\sum_{z\in Z_v^{d-1}}\inf_{y\in(mz+\Gamma_m)\cap T\cap X}\mu(x+y+A) = \infty.$$

Therefore (2.14) holds for this A.

If A satisfies (2.16) then it follows by almost the same argument that (2.15) holds.

Next we consider the results in the converse direction. Let A be a Borel subset of X. Suppose there is a sufficiently large $T \in \mathscr{T}_v$ and a compact subset C of X such that

$$\sum_T \inf_{y \in C} \mu(n+y+A) < \infty.$$

Then

$$\sum_{z \in Z_v^{d-1}} \sum_{(mz+\Gamma_m) \cap T} \inf_{y \in C} \mu(n+y+A) < \infty.$$

Thus there exist $n_z \in (mz+\Gamma_m) \cap T \cap Z^d$ and $y_z \in C$ such that

$$\sum_{z \in Z_v^{d-1}} \mu(n_z+y_z+A) < \infty.$$

Let ν be defined by

$$\nu(B) = \sum_{z \in Z_v^{d-1}} 1_B(-n_z-y_z).$$

Then $\nu \in \mathscr{N}_{v,\lambda}$ for some $0<\lambda<\infty$ and

$$(\nu * \mu)(A) = \sum_{z \in Z_v^{d-1}} \mu(n_z+y_z+A) < \infty.$$

This completes the proof of Theorem 3.

Proof of Theorem 4(i). As in the proof of Theorem 3(i), it follows from the assumptions that

$$\lim_{x \cdot v \to \infty} \sum_{z \in Z_v^{d-1}} \inf_{y \in (mz+\Gamma_m) \cap T \cap X} \mu(x+y+A) = \infty.$$

Let $a_z(x)$, $z \in Z_v^{d-1}$ and $x \in X$, be such that

$$0 \leqq a_z(x) \leqq \inf_{y \in (mz+\Gamma_m) \cap T \cap X} \mu(x+y+A),$$

$$\sum_{z \in Z_v^{d-1}} a_z(x) < \infty,$$

and

$$\lim_{x \cdot v \to \infty} \sum_{z \in Z_v^{d-1}} a_z(x) = \infty.$$

We can assume that N is supported by T. Then

$$\begin{aligned}(N * \mu)(x+A) &= \sum_{z \in Z_v^{d-1}} \int_{mz+\Gamma_m} N(-dy)\mu(x+y+A) \\ &\geqq \sum_{z \in Z_v^{d-1}} a_z(x) N(-(mz+\Gamma_m)).\end{aligned}$$

Choose $\varepsilon>0$. By (2.8) we can find $M>0$ and $m \geqq 1$ such that

$$E[(N(-(mz+\Gamma_m))-M)^-] \leqq \varepsilon M,$$

where $r^- = -r$ if $r<0$ and $r^- = 0$ if $r \geqq 0$. It follows that

$$E\left[\left(\sum_{z\in Z_v^{d-1}} a_z(x)N(-(mz+\Gamma_m))-M\sum_{z\in Z_v^{d-1}} a_z(x)\right)^-\right] \leqq \varepsilon M \sum_{z\in Z_v^{d-1}} a_z(x).$$

Thus for $0<y<\infty$

$$P\left\{\sum_{z\in Z_v^{d-1}} a_z(x)N(-(mz+\Gamma_m)) \leqq y\right\} \leqq \frac{\varepsilon M \sum_{z\in Z_v^{d-1}} a_z(x)}{M\sum_{z\in Z_v^{d-1}} a_z(x)-y}.$$

Consequently

$$\lim_{x\cdot v\to\infty} P\left\{\sum_{z\in Z_v^{d-1}} a_z(x)N(-(mz+\Gamma_m)) \leqq y\right\} = 0$$

and hence

$$\lim_{x\cdot v\to\infty} \sum_{z\in Z_v^{d-1}} a_z(x)N(-(mz+\Gamma_m)) \overset{\mathrm{P}}{=} \infty.$$

Therefore

$$\lim_{x\cdot v\to\infty} (N*\mu)(x+A) \overset{\mathrm{P}}{=} \infty.$$

Proof of Theorem 4(ii). The proof of (ii) is similar to that of (i), but we give the details since it is (ii) which is needed in Stone [3].

Let C be a compact subset of X. It follows from the assumptions of (ii) that

$$\sum_{z\in Z_v^{d-1}} \inf_{y\in X\cap(mz+\Gamma_m)\cap T\cap X} \inf_{x\in C} \mu(x+y+A) = \infty.$$

Set

$$a_z = \inf_{y\in X\cap(mx+\Gamma_m)\cap T\cap X} \inf_{y\in C} \mu(x+y+A).$$

Then

$$(N*\mu)(x+A) \geqq \sum_{z\in Z_v^{d-1}} a_z N(-(mz+\Gamma_m)), \qquad x\in C.$$

Choose $a_z^{(n)}$, $z\in Z_v^{d-1}$ and $n\geqq 1$, such that

$$0 \leqq a_z^{(n)} \leqq a_z, \qquad \sum_{z\in Z_v^{d-1}} a_z^{(n)} < \infty,$$

and

$$\lim_{n\to\infty} \sum_{z\in Z_v^{d-1}} a_z^{(n)} = \infty.$$

Choose $\varepsilon>0$ and let $M>0$ and $m\geqq 1$ be as in the proof of (i). Then, as in (i), we observe that

$$P\left\{\sum_{z\in Z_v^{d-1}} a_z N(-(mz+\Gamma_m)) \leqq y\right\} \leqq \frac{\varepsilon M \sum_{z\in Z_v^{d-1}} a_z^{(n)}}{M\sum_{z\in Z_v^{d-1}} a_z^{(n)}-y}$$

and hence

$$P\left\{\sum_{z\in Z_v^{d-1}} a_z N(-(mz+\Gamma_m)) \leqq y\right\} \leqq \varepsilon.$$

Since ε can be made arbitrarily small and y can be made arbitrarily large, it follows that

$$P\left\{\sum_{z\in Z_v^{d-1}} a_z N(-(mz+\Gamma_m)) = \infty\right\} = 1$$

and therefore that

$$P\{(N*\mu)(x+A) = \infty \text{ for } x\in C\} = 1.$$

Since X is σ-compact, we have (2.17) as desired.

References

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